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A HYBRID METHOD FOR THE SOLUTION OF SOME MULTI-COMMODITY SPATIA--ETC(U)
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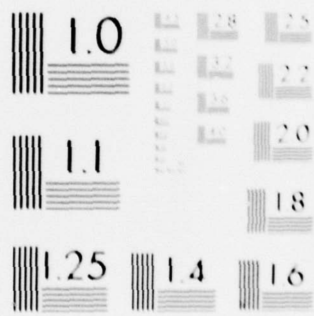


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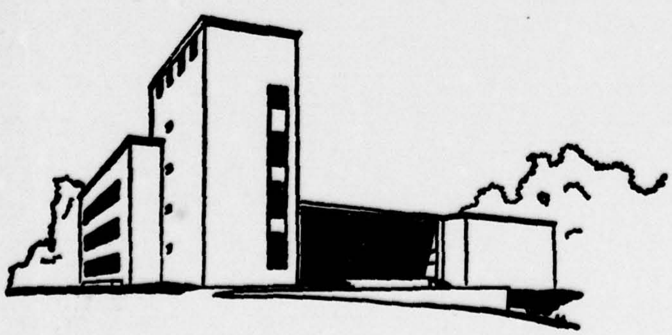

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(6) A HYBRID METHOD FOR THE SOLUTION OF SOME MULTI-COMMODITY SPATIAL EQUILIBRIUM PROBLEMS.

by

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# A HYBRID METHOD FOR THE SOLUTION OF SOME MULTI-COMMODITY SPATIAL EQUILIBRIUM PROBLEMS

by

Jong-Shi Pang

ABSTRACT. In this paper, we first derive a unified formulation of the multi-commodity transportation and transshipment spatial equilibrium models as a linear complementarity problem with certain block structure. We then show that a block successive overrelaxation method is applicable for solving the resulting complementarity problem. The method consists of solving a sequence of subproblems of the single-commodity type. These subproblems are to be solved by a special-purpose principal pivoting algorithm developed in an earlier paper. Computational experience of solving some fairly large problems by the proposed hybrid method is presented.

Key Words. Successive overrelaxation, Spatial equilibrium, Transportation, Transshipment, Linear complementarity, Special structure, Computational experience.

## 1. INTRODUCTION

In the past few years, there have been several published studies on the numerical solution of various economic spatial equilibrium models [1, 3, 5, 7, 9]. Basically, such models are concerned with the determination of the regional production, consumption and inter-regional flows of several commodities so as to achieve price equilibrium. These models have important practical applications, see [4, 6, 9, 11] and references therein.

The objective of this paper is to propose a hybrid method for solving two types of multi-commodity spatial equilibrium models; namely the transportation and transshipment models. These models can be viewed as extensions of their linear analogs which are the well-known linear transportation and transshipment models. The method operates by solving a sequence of subproblems of the single-commodity type. These subproblems are to be solved by a special-purpose principal pivoting algorithm described in an earlier paper [7].

It is a rather well-known fact that most spatial equilibrium models can be formulated as complementarity problems, see [8] e.g. With the assumption of linear supply and demand functions,[1/] these become linear complementarity problems. Under further assumptions on the supply and demand functions, the latter complementarity problems can in principle, be solved by a number of pivoting methods (most notably, Lemke's almost complementary pivoting algorithm). However, due to the fact that typically, these linear complementarity problems are enourmously large,

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1/ Such an assumption is made very often in practice, see [9, 11].

it is rather impractical, if not impossible to apply the pivoting methods in a straightforward manner.

Among the recently proposed methods for solving multi-commodity spatial equilibrium problems the most promising approach seems to be the Bender's decomposition proposed in [9, 10]. Nevertheless, despite the rather encouraging computational results reported in the reference for solving various practical applications, there is a basic drawback of the approach, namely, it requires solving quadratic subprograms (in addition to solving linear ones) which could still be fairly large.

In essence, the hybrid method proposed is a specialization of the block successive overrelaxation (SOR) iterative method (described in [2]) applied to a linear complementarity problem with certain block structure. As we shall demonstrate, such a complementarity problem provides a unified formulation for both the transportation and transshipment spatial equilibrium models.

## 2. THE TWO SPATIAL EQUILIBRIUM MODELS

We first describe the transportation spatial equilibrium model. A certain number m of commodities are to be shipped from $n_1$ supply locations to $n_2$ demand locations. The problem is to determine a set of flow variables $X = (X_{\alpha\beta i})$, supply quantities $x = (x_{\alpha i})$, demand quantities $y = (y_{\beta i})$, optimal market supply prices $\lambda^1 = (\lambda^1_{\alpha i})$ and optimal market demand prices $\lambda^2 = (\lambda^2_{\beta i})$ so that the equilibrium conditions below are satisfied for each $\alpha = 1,\ldots,n_1$, $\beta = 1,\ldots,n_2$, and $i = 1,\ldots,m$:

(a) Linear regional supply and demand functions

$$p^1_{\alpha i} = s_{\alpha i} + \sum_{k=1}^{m} g_{\alpha i k}\, x_{\alpha k} \quad , \quad p^2_{\beta i} = -d_{\beta i} - \sum_{k=1}^{m} h_{\beta i k}\, y_{\beta k}$$

(b) Optimum production and consumption

$$p^1_{\alpha i} \geq \lambda^1_{\alpha i} \quad , \quad x_{\alpha i} \geq 0 \qquad (p^1_{\alpha i} - \lambda^1_{\alpha i})x_{\alpha i} = 0$$

$$p^2_{\beta i} \leq \lambda^2_{\beta i} \qquad y_{\beta i} \geq 0 \qquad (\lambda^2_{\beta i} - p^2_{\beta i})y_{\beta i} = 0$$

(c) Optimum excess supply and demand

$$x_{\alpha i} \geq \sum_{\gamma=1}^{n_2} X_{\alpha\gamma i} \quad , \quad \lambda^1_{\alpha i} \geq 0 \qquad \left(\sum_{\gamma=1}^{n_2} X_{\alpha\gamma i} - x_{\alpha i}\right)\lambda^1_{\alpha i} = 0$$

$$y_{\beta i} \leq \sum_{\gamma=1}^{n_1} X_{\gamma\beta i} \quad , \quad \lambda^2_{\beta i} \geq 0 \qquad \left(\sum_{\gamma=1}^{n_1} X_{\gamma\beta i} - y_{\beta i}\right)\lambda^2_{\beta i} = 0$$

(d) Optimum spatial allocation

$$c_{\alpha\beta i} + \lambda^1_{\alpha i} - \lambda^2_{\beta i} \geqq 0 \quad , \quad X_{\alpha\beta i} \geqq 0 \quad , \quad (c_{\alpha\beta i} + \lambda^1_{\alpha i} - \lambda^2_{\beta i})X_{\alpha\beta i} = 0 .$$

In condition (a) above, $s_{\alpha i}$ , $d_{\beta i}$, $g_{\alpha ik}$ and $h_{\beta ik}$ are given constants defining the supply and demand functions. In condition (d), $c_{\alpha\beta i}$ is the unit transportation cost from supply $\alpha$ to demand $\beta$ . More details on the description of this model can be found in [11].

By using condition (a) to eliminate the variables $p^1_{\alpha i}$ and $p^2_{\beta i}$ , we may formulate the model as the linear complementarity problem: find vectors x, y, X, $\lambda^1$ and $\lambda^2$ such that

$$\begin{array}{llllllll}
(2.1) & u = s + Gx & & -\lambda^1 & & \geqq 0 & x \geqq 0 \\
& v = d & + Hy & & +\lambda^2 & \geqq 0 & y \geqq 0 \\
& Y = c & & + S^T\lambda^1 & - D^T\lambda^2 & \geqq 0 & X \geqq 0 \\
& \mu^1 = \quad x & & - SX & & \geqq 0 & \lambda^1 \geqq 0 \\
& \mu^2 = & - y & + DX & & \geqq 0 & \lambda^2 \geqq 0
\end{array}$$

$$u^T x = v^T y = Y^T X = (\mu^1)^T \lambda^1 = (\mu^2)^T \lambda^2 = 0 .$$

Here $\begin{pmatrix} S \\ D \end{pmatrix}$ is the multi-commodity transportation constraint matrix, i.e.,

(2.2a)
$$S = \left[\begin{array}{c|c|c|c} \underbrace{I_m \; - - - \; I_m}_{n_2} & & & \\ & \underbrace{I_m \; - - - \; I_m}_{n_2} & & \\ & & \ddots & \\ & & & \underbrace{I_m \; - - - \; I_m}_{n_2} \end{array}\right] \left.\vphantom{\begin{array}{c}1\\1\\1\\1\\1\end{array}}\right\} n_1$$

(2.2b)
$$D = \underbrace{\left[\begin{array}{c|c|c|c} \underbrace{\begin{matrix} I_m & & \\ & \ddots & \\ & & I_m \end{matrix}}_{n_2} & \underbrace{\begin{matrix} I_m & & \\ & \ddots & \\ & & I_m \end{matrix}}_{n_2} & - - - & \underbrace{\begin{matrix} I_m & & \\ & \ddots & \\ & & I_m \end{matrix}}_{n_2} \end{array}\right]}_{n_1}$$

with $I_k$ denoting the identity matrix of order k; G and H are the block diagonal matrices

(2.2c)
$$G = \begin{pmatrix} G_1 & & \\ & \ddots & \\ & & G_{n_1} \end{pmatrix}, \quad H = \begin{pmatrix} H_1 & & \\ & \ddots & \\ & & H_{n_2} \end{pmatrix}$$

where $G_\alpha = (g_{\alpha i j})$ and $H_\beta = (H_{\beta i j})$ are m by m matrices.

Basically, the <u>transshipment spatial equilibrium model</u> is defined by a set of equilibrium conditions similar to (a) - (d) above. In this model, the commodities, instead of being shipped directly from the supply to the demand locations, may go through some intermediate transshipment points. If n is the total number of regions in the market, then the problem is to find a set of flow variables $X = (X_{\alpha\beta i})$, net flow quantities $y = (y_{\alpha i})$ so that the equilibrium conditions below are satisfied for each $\alpha, \beta = 1,\ldots,n$ and $i = 1,\ldots,m$:

(A) Linear regional supply and demand functions

$$p_{\alpha i} = a_{\alpha i} - \sum_{k=1}^{m} b_{\alpha i k}\, y_{\alpha k}$$

(B) Conservation of flows

$$y_{\alpha i} = \sum_{\gamma=1}^{n} X_{\gamma\alpha i} - \sum_{\gamma=1}^{n} X_{\alpha\gamma i}$$

(C) Optimum spatial allocation

$$c_{\alpha\beta i} + p_{\alpha i} - p_{\beta i} \geqq 0 \quad , \quad X_{\alpha\beta i} \geqq 0 \quad , \quad (c_{\alpha\beta i} + p_{\alpha i} - p_{\beta i})X_{\alpha\beta i} = 0 \; .$$

Here, we have assumed, for the sake of simplicity, that there is a route joining any two regions. For more details on the description of this model, see [8].

By using conditions (A) and (B) to eliminate the variables $y_{\alpha i}$ and $p_{\alpha i}$, we obtain the linear complementarity problem formulation of the model: find a vector X such that

(2.3) $$Y = c + P^T a + P^T B P X \geqq 0 \quad , \quad X \geqq 0 \quad \text{and} \quad X^T Y = 0 \; .$$

Here P is the multi-commodity node-arc incidence matrix[2/] of a simple, complete digraph G with n nodes and B is the block diagonal matrix

$$(2.4) \qquad B = \begin{pmatrix} B_1 & & \\ & \ddots & \\ & & B_n \end{pmatrix}$$

where each $B_\alpha = (b_{\alpha i j})$ is m by m. See [7] for more details on the derivation of the problem (2.3).

2/ It is obtained by replacing the ones in an ordinary node-arc incidence matrix by identity matrices of order m. (cf. (2.2a) and (2.2b).)

## 3. A UNIFIED FORMULATION

In the formulation (2.1) and (2.3), the vector X is arranged according to routes between regions; i.e., commodities passing through the same route are ordered consecutively. For our purpose, it would be more convenient to reformulate the problems with the vector X partitioned according to commodities. For the transshipment problem (2.3), this can be easily done. In fact, the resulting reformulation has the form

(3.1) $$\tilde{Y} = \tilde{c} + \tilde{P}^T a + \tilde{P}^T \tilde{A} \tilde{P} \tilde{X} \geqq 0 \quad , \quad \tilde{X} \geqq 0 \quad \text{and} \quad \tilde{Y}^T \tilde{X} = 0 \ .$$

Here $\tilde{P}$ is the block diagonal matrix

(3.2a) $$\tilde{P} = \begin{pmatrix} P_1 & & \\ & \ddots & \\ & & P_m \end{pmatrix}$$

where each $P_i$ is the (single-commodity) node-arc incidence matrix of a simple, complete digraph with n nodes and $\tilde{A}$ is the block matrix

(3.2b) $$\tilde{A} = \begin{pmatrix} A_{11} & \cdots & A_{1m} \\ \vdots & & \vdots \\ A_{m1} & \cdots & A_{mm} \end{pmatrix}$$

where $A_{ij}$ is the n by n diagonal matrix whose $\alpha$-th diagonal entry is $b_{\alpha i j}$. The vectors $\tilde{c}$, $\tilde{Y}$ and $\tilde{X}$ are the corresponding rearrangements of c, Y and X respectively.

It is also easy to reformulate the problem (2.1) in terms of commodities. However, it is our contention that the problem can be cast in the form (3.1) as well. In what follows, we show how this is done under three assumptions:

(1) Both matrices S and D are nonnegative

(2) Each column of D has at most one nonzero entry

(3) The vector c is strictly positive.

Observe that the matrices S and D are not required to have the transportation structure (2.2a, b) to satisfy the first two conditions.

Lemma 1. Suppose that the assumptions (1) - (3) are satisfied. Then any complementary solution to (2.1) must satisfy the condition: $\mu^2 = 0$.

Proof. Suppose that $(x, y, X, \lambda^1, \lambda^2)$ is a complementary solution to (2.1) with $\mu_i^2 > 0$ for some component i. Then by complementarity, it follows that $\lambda_i^2 = 0$. Let

$$J = \{j: D_{ij} \neq 0\} .$$

Then for each $j \in J$, we have

$$Y_j = c_j + (S^T\lambda^1)_j - D_{ij}\lambda_i^2 = c_j + (S^T\lambda^1)_j > 0$$

which implies that $X_j = 0$. Hence,

$$0 < \mu_i^2 = - y_i + \sum_{j \in J} D_{ij}X_j = - y_i \leqq 0$$

which is a contradiction. Q.E.D.

Referring to the transportation spatial model, Lemma 1 says that if the transportation costs are strictly positive, then no excess demand is possible at equilibrium.

Proposition 2. Under the same assumptions, problem (2.1) is equivalent to

$$(3.3)\qquad \begin{array}{llllll} u & = s & + Gx & - \lambda & \geqq 0 & x \geqq 0 \\ Y & = (c + D^T d) & + D^T H D X & + S^T \lambda & \geqq 0 & X \geqq 0 \\ \mu & = & x & - S X & \geqq 0 & \lambda \geqq 0 \end{array}$$

$$u^T x = Y^T X = \mu^T \lambda = 0 \ .$$

in the sense that a complementary solution to one problem will always give rise to one such solution of the other.

Proof. Let $(x, y, X, \lambda^1, \lambda^2)$ be a complementary solution to (2.1). Then by Lemma 1, we have $y = DX$. Hence

$$Y = c + S^T \lambda^1 - D^T (v - d - Hy)$$

or equivalently,

$$(3.4)\qquad Y + D^T v = (c + D^T d) + D^T H D X + S^T \lambda^1 \ .$$

We claim that $(x, X, \lambda^1)$ is a complementary solution to (3.3). It suffices to show

$$X^T (c + D^T d + D^T H D X + S^T \lambda^1) = 0.$$

According to (3.4), the left-hand term is equal to

$$X^T (Y + D^T v) = (DX)^T v = y^T v = 0 \ .$$

Conversely, suppose that $(x, X, \lambda)$ is a complementary solution to (3.3). Define

$$\lambda^1 = \lambda \quad , \quad y = DX \quad \text{and} \quad \lambda^2 = (d + Hy)^-$$

where $z^-$ is the negative part of the vector $z$. We claim that $(x, y, X, \lambda^1, \lambda^2)$ is a complementary solution to (2.1). To show that it is feasible, we have

$$d + Hy + \lambda^2 = (d + Hy)^+ \geqq 0$$

where $z^+$ is the positive part of the vector $z$. Moreover, it is obvious that the following holds

$$(3.5) \qquad c + S^T\lambda^1 - D^T\lambda^2 = (c + D^Td) + D^THDX + S^T\lambda^1 - D^T(d + Hy)^+ \ .$$

If $(c + S^T\lambda^1 - D^T\lambda^2)_j < 0$ for some component $j$, then column $j$ of $D$ must contain exactly one positive entry, say $D_{ij} > 0$. Furthermore, we must have $(d + Hy)_i > 0$ implying $\lambda_i^2 = 0$. Hence

$$0 > (c + S^T\lambda^1 - D^T\lambda^2)_j = (c + S^T\lambda^1)_j - D_{ij}\lambda_i^2 \geqq 0$$

which is impossible. Therefore the vector $(x, y, X, \lambda^1, \lambda^2)$ is feasible to (2.1). To complete the proof, it remains to show that

$$y^T(d + Hy)^+ = X^T(c + S^T\lambda^1 - D^T\lambda^2) = 0 \ .$$

Suppose that $(d + Hy)_i > 0$. It then follows that $\lambda_i^2 = 0$ . By using an argument similar to the proof of Lemma 1, we may readily deduce that $y_i = 0$. Consequently, it holds that $y^T(d + Hy)^+ = 0$. Finally, it follows from (3.5) that

$$X^T(c + S^T\lambda^1 - D^T\lambda^2) = X^T[(c + D^Td) + D^THDX + S^T\lambda^1] - (DX)^T(d + Hy)^+$$
$$= - y^T(d + Hy)^+ = 0$$

by what has just been proven. Q.E.D.

Remark. The second part of the proof remains valid if c is merely nonnegative.

The last proposition establishes a first-stage transformation of the problem (2.1). To complete the process, we prove

Lemma 3. If the matrix S is nonnegative, then any complementary solution to (3.3) must satisfy: $u^T(SX) = u^T\mu = 0$.

Proof. In fact, if $u_i > 0$, then $x_i = 0$ by complementarity. Hence

$$0 \leqq \mu_i = -(SX)_i \leqq 0$$

from which the desired equalities follow readily. Q.E.D.

Proposition 4. Suppose that assumptions (1) - (3) are satisfied. Then problem (3.3) (and thus (2.1)) is equivalent to

$$(3.6) \qquad \begin{aligned} Y &= (c + D^T d + S^T s) + (D^T HD + S^T GS)X + S^T G\mu \geqq 0 \quad , \quad X \geqq 0 \\ \lambda &= s + GSX + G\mu \geqq 0 \quad , \quad \mu \geqq 0 \\ Y^T X &= \lambda^T \mu = 0 \end{aligned}$$

in the sense specified in Proposition 2. The latter linear complementarity problem can be written in the form

$$w = r + P^T a + P^T BPz \geqq 0 \quad , \quad z \geqq 0 \quad w^T z = 0$$

where

$$r = \begin{pmatrix} c \\ 0 \end{pmatrix}, \quad a = \begin{pmatrix} d \\ s \end{pmatrix}, \quad w = \begin{pmatrix} Y \\ \lambda \end{pmatrix}, \quad z = \begin{pmatrix} X \\ \mu \end{pmatrix}$$

$$P = \begin{pmatrix} D & 0 \\ S & I \end{pmatrix} \quad \text{and} \quad B = \begin{pmatrix} H & 0 \\ 0 & G \end{pmatrix}$$

<u>Proof</u>. Suppose that $(x, X, \lambda)$ is a complementary solution to (3.3). We claim that $(X, \mu)$ is a complementary solution to (3.6). In fact, we have

$$s + GSX + G\mu = s + Gx = u + \lambda \geqq 0$$

and

$$(c + D^T d + S^T s) + (D^T HD + S^T GS)X + S^T G\mu$$

$$= (c + D^T d) + D^T HDX + S^T \lambda + S^T u \geqq 0$$

establishing the feasibility. The desired complementarity relationships follow from Lemma 3.

Conversely, let $(X, \mu)$ be a complementary solution to (3.6). Define

$$x = \mu + SX \quad \text{and} \quad \lambda = s + Gx .$$

Obviously $(x, X, \lambda)$ solves (3.3) . Q.E.D.

The last proposition shows that if the transportation costs are all positive, then the transportation spatial equilibrium problem (2.1) can be cast as a linear complementarity problem of the same type as the one obtained from the transshipment model. In particular, by partitioning the flow vector (X) and the excess supply vector ($\mu$) according to commodities, we may rewrite the linear complementarity problem (3.6) in the form (3.1) and (3.2) with each $P_i$ and $A_{ij}$ given by

$$(3.7) \qquad P_i = \begin{pmatrix} D_i & 0 \\ S_i & I \end{pmatrix} \quad , \qquad A_{ij} = \begin{pmatrix} A^1_{ij} & 0 \\ 0 & A^2_{ij} \end{pmatrix}$$

where $A^1_{ij}$ and $A^2_{ij}$ are, respectively, $n_2$ by $n_2$ and $n_1$ by $n_1$ diagonal matrices with $(A^1_{ij})_{\alpha\alpha} = h_{\alpha ij}$ and $(A^2_{ij})_{\beta\beta} = g_{\beta ij}$; and where $\begin{pmatrix} S_i \\ D_i \end{pmatrix}$ is the single-commodity, transportation constraint matrix, i.e., (2.2a) and (2.2b) with $m = 1$.

It is interesting to note that in both cases, the matrices $A_{ij}$ are diagonal and the $P_i$ are identical. In fact, each $P_i$ is the node-arc incidence matrix of the graph (or network) underlying the model. This is clear in the transshipment case. In the other case, the $P_i$ (cf. (3.7)) can be considered as the node-arc incidence of a transportation bipartite graph with a self-loop connecting each supply node. Such a loop represents the excess supply at that node.

To summarize, we conclude that both the multi-commodity transportation and transshipment spatial models can be formulated as a linear complementarity problem of the form (note the change of notation)

$$(3.8) \qquad w = r + P^T s + P^T APz \geq 0 \quad , \qquad z \geq 0 \quad , \qquad w^T z = 0$$

where the matrix P is block diagonal with the number of blocks equal to the number of commodities. We point out that the matrix $P^TAP$ has the block structure

$$(3.9) \qquad P^TAP = \begin{pmatrix} P_1^T A_{11} P_1 & \cdots & P_1^T A_{1m} P_m \\ \vdots & & \vdots \\ P_m^T A_{m1} P_1 & \cdots & P_m^T A_{mm} P_m \end{pmatrix} .$$

## 4. THE HYBRID METHOD

There are two basic reasons motivating the proposal of a hybrid method for solving the two spatial models described earlier. The first is the large size of the resulting linear complementarity problem which is typical in practice. For instance, a transshipment problem with 10-commodities and 50-regions would give rise to a 24,500 by 24,500 matrix in the complementarity problem. The second reason is due to the recent development of an efficient special-purpose algorithm for solving the single-commodity models[3/] (see [7]). It is hoped that this algorithm would provide an effective subroutine for solving the multi-commodity problems.

To state the hybrid method, we refer to the unified formulation of the two models, i.e., the linear complementarity problem (3.8) and (3.9). Recall also that the matrix P is block diagonal with diagonal blocks $P_i$ .

Algorithm

Step 0. Let $z^0 = (z_1^0,\dots,z_m^0)$ be an arbitrary nonnegative vector and let $\bar{\omega}$ be a scalar with $0 < \bar{\omega} < 2$. Put $k = 0$ .

Step 1. For $i = 1,\dots,m$, solve the linear complementarity problem

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3/ Although the reference treats only the single-commodity transshipment model, due to the fact that the transportation model gives rise to exactly the same type of linear complementarity problem it is not difficult to extend the algorithm described in the reference to handle the transportation model. See section 5 for computational results on this extension.

(4.1) $$w_i = r_i + P_i^T s_i^k + P_i^T A_{ii} P_i z_i \geqq 0 \quad , \quad z_i \geqq 0 \text{ and } w_i^T z_i = 0$$

with[4/] $s_i^k = s_i + \sum_{j<i} A_{ij} P_j z_j^{k+1} + \sum_{j>i} A_{ij} P_j z_j^k$ ; let $z_i^{k+\frac{1}{2}}$ be a solution and set

$$z_i^{k+1} = z_i^k + \omega_i^{k+1}(z_i^{k+\frac{1}{2}} - z_i^k)$$

where $\omega_i^{k+1} = \max\{\omega : \omega \leqq \bar{\omega} \; , \; z_i^{k+1} \geqq 0\}$ .

Step 2. Let

$$J = \{(i,j) : (z_i^{k+1})_j > 0 \text{ , or } (w_i^{k+1})_j = (r_i + P_i^T(s_i + \sum_{\ell=1}^m A_{i\ell} P_\ell z_\ell^{k+1}))_j < 0\}$$

If $\max_{(i,j) \in J} |(w_i^{k+1})_j| \leqq \epsilon$ where $\epsilon$ is a given positive tolerance, stop. An approximate solution is at hand. Otherwise return to Step 1 with k replaced by k+1.

In essence, the algorithm is a specialization of the modified block SOR method for solving a general linear complementarity problem with block structure (see [2]). Concerning its convergence, we state the following principal result.

Theorem 5. Suppose that the matrix A is symmetric positive definite. Suppose also that for i = 1,...,m, the system

(4.2) $$0 \leqq y_i \neq 0 \quad , \quad P_i y_i = 0 \quad , \quad r_i^T y_i \leqq 0$$

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4/ The vectors r and s are partitioned as $r = (r_1^T, \ldots, r_m^T)^T$ and $s = (s_1^T, \ldots, s_m^T)^T$ in accordance with the matrix P.

is inconsistent. Then the sequence $\{z^k = (z_1^k,\dots,z_m^k)\}$ of vectors generated by the algorithm has a limit point. Moreover, any such limit point solves the linear complementarity problem (3.8).

It should be pointed out that even though the matrix A is positive definite, the matrix $P^TAP$ (and each of its diagonal submatrices $P_i^TA_{ii}P_i$) may not be so. Hence, the theorem is not a direct consequence of the convergence result established in the reference. Nevertheless, it can be proved in very much the same way. In what follows, we sketch the proof.

First of all, the inconsistency of the systems (4.2) and the positive definiteness of the matrix A imply the feasibility and hence the solvability of each of the subproblems (4.1). Thus each vector $z_i^{k+\frac{1}{2}}$ is well-defined. Next, by using the function $f(z) = (r + P^Ts)^Tz + \frac{1}{2}(Pz)^TAPz$ to monitor the progress of the algorithm, it is not difficult to show that the sequence $\{f(z^k)\}$ is monotonically decreasing. From this it follows that the sequence $\{z^k\}$ must be bounded and therefore has a limit point. Finally, that any such limit point solves the linear complementarity problem (3.8) can be proved by using the positive definiteness of the matrix A.

Without the positive definiteness of the matrix $P^TAP$, there is in general no guarantee that the entire sequence $\{z^k\}$ will in fact converge. However, the next result shows that the sequence $\{Pz^k\}$ will.

<u>Proposition 6</u>. Under the assumptions of Theorem 5, the sequence $\{Pz^k\}$ converges to $Pz^*$ where $z^*$ is a limit point of $\{z^k\}$.

<u>**Proof**</u>. Observe that any limit point of the sequence $\{Pz^k\}$ must be of

the form $Pz^*$ where $z^*$ is a limit point of the sequence $\{z^k\}$ . The fact that $\{Pz^k\}$ has at least one limit point follows from its boundedness (which is implied by the boundedness of $\{z^k\}$). Suppose now that $\{Pz^k\}$ has another limit point given by $Py^*$ where $y^*$ is some limit point of $\{z^k\}$ . By Theorem 5, both $y^*$ and $z^*$ solve the linear complementarity problem (3.8). By the positive definiteness of A, it can be proved easily that $Py^* = Pz^*$. Thus, the sequence $\{Pz^k\}$ has a unique limit point and therefore converges. Q.E.D.

Specializing Theorem 5 to the two spatial models, we obtain

Theorem 7. Suppose that each of the matrices $G_\alpha$, $H_\beta$ and $B_\alpha$ (cf. (2.2c) and (2.4)) is symmetric positive definite. Suppose also that the vector c of transportation costs is positive (in both models). Then the assumptions of Theorem 5 are satisfied. In particular, the sequence of vectors generated by the algorithm will converge, in the sense specified in Theorem 5 and Proposition 6, to a solution of the two models.

The proof of Theorem 7 is easy and thus omitted. Observe that in the specialization, each subproblem (4.1) is precisely one of the single-commodity type to which the algorithm described in [7] can be applied.

## 5. COMPUTATIONAL EXPERIENCE

In this section, we report our computational experience[5/] with the hybrid method for solving some randomly generated spatial transportation and transshipment models. For completeness, we first present a brief summary of the computational results (see Tables 1 and 2) with the special-purpose principal pivoting algorithm (described in [7]) for solving the single-commodity problems. We explain how the data were generated. For the transportation model (cf. the formulation (3.6)), the transportation costs were randomly generated in the interval (0,10), the components of the vectors s and d in ( - 50,0) and ( - 10,0) respectively and the diagonal entries of the matrices G and H in (0,5). The data for the transshipment model were generated in the same way as in [7].

The third column (basic arcs) in Tables 1 and 2 refers to the number of flow variables that are positive in the final solution. Notice that this number is bounded above by the total number of regions less one. The fourth column (excess supplies) in Table 1 refers to the number of supply regions where the actual supply quantity exceeds the effective supply shipped.

As with all relaxation methods, the efficiency of the hybrid method proposed is critically dependent on the choice of the parameter $\bar{\omega}$. In the experimentation, we have tried several values. In Tables 3 and 4 below, we report a summary of the results with the use of the particular

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5/ All the computations were performed on a DEC-20 computer and in double precision. The computer codes were written in FORTRAN. All the timings reported are exclusive of input and output.

| (#supplies, #demands) | #arcs | #basic arcs | #excess supplies | #pivots | CPU time (in sec.) Total | Per pivot |
|---|---|---|---|---|---|---|
| (80,80) | 6,400 | 100 | 57 | 794 | 298.983 | 0.389 |
| (80,40) | 3,200 | 55 | 65 | 399 | 77.307 | 0.194 |
| (60,120) | 7,200 | 138 | 38 | 933 | 406.102 | 0.435 |
| (60,100) | 6,000 | 129 | 27 | 707 | 250.170 | 0.354 |
| (40,120) | 4,800 | 148 | 4 | 603 | 180.451 | 0.299 |
| (20,100) | 2,000 | 96 | 1 | 192 | 26.388 | 0.137 |

Table 1. Single-commodity transportation spatial model

| #regions | #arcs | #basic arcs | #pivots | CPU time (in sec.) Total | per pivot |
|---|---|---|---|---|---|
| 120 | 14,280 | 71 | 377 | 158.397 | 0.420 |
| 105 | 10,920 | 62 | 142 | 45.527 | 0.321 |
| 90 | 8,010 | 59 | 311 | 76.272 | 0.249 |
| 75 | 4,150 | 43 | 175 | 29.232 | 0.167 |
| 30 | 870 | 17 | 35 | .945 | 0.027 |

Table 2. Single-commodity transshipment spatial model

$\bar{\omega}$ giving the fastest convergence. The data for the problems solved were generated in the same way as in the single-commodity case. The tolerance $\epsilon$ for the termination criteria (cf. Step 2 in the algorithm) was chosen to be $10^{-5}$. As the task of testing for termination in each iteration is rather time-consuming (because of the large number of variables in the problem), we chose not to perform such test at every iteration, but rather at every 5 iterations. The entries in the two tables below should be self-explanatory.

| #supplies | #demands | #commodities | $\bar{\omega}$ | #iterations | total CPU time (sec.) |
|---|---|---|---|---|---|
| 20 | 20 | 3 | 1.4 | 30 | 264.871 |
| 15 | 15 | 9 | 1.4 | 35 | 309.769 |
| 15 | 15 | 6 | 1.4 | 55 | 371.865 |
| 10 | 10 | 9 | 1.4 | 35 | 98.998 |
| 15 | 15 | 3 | 1.2 | 15 | 56.487 |
| 10 | 10 | 6 | 1.2 | 30 | 62.253 |

Table 3. Multi-Commodity transportation spatial model

| #regions | #commodities | $\bar{\omega}$ | #iterations | Total CPU time (sec.) |
|---|---|---|---|---|
| 40 | 2 | 1.7 | 140 | 592.419 |
| 20 | 10 | 1.7 | 125 | 326.102 |
| 20 | 6 | 1.7 | 120 | 150.913 |
| 15 | 10 | 1.7 | 60 | 56.449 |
| 15 | 4 | 1.7 | 45 | 18.355 |
| 10 | 10 | 1.75 | 85 | 29.799 |
| 10 | 8 | 1.75 | 35 | 8.886 |

Table 4. Multi-commodity transshipment spatial model

## 6. CONCLUSION

In this paper, we have shown how a hybrid iterative method can be used to solve the multi-commodity transportation and transshipment spatial equilibrium models. At this stage, we are unable to compare the numerical performance of the proposed method with other methods that are applicable. To the best of our knowledge, there is no published numerical results for solving problems as large as the ones solved in this paper. (The largest problem solved by the Bender's decomposition approach has only eight supply and demand regions and four commodities. See [9,10].) It would be interesting to see how these problems can actually be solved by other methods, especially the Bender's approach.

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~~In this paper, we first derive~~ a unified formulation of the multi-commodity transportation and transshipment spatial equilibrium models HAS BEEN DEVISED as a linear complementarity problem with certain block structure. ~~We then show~~ IT IS SHOWN that a block successive overrelaxation method is applicable for solving the resulting complementarity problem. The method consists of solving a sequence of subproblems of the single-commodity type. These subproblems are to be solved by a special-purpose principal pivoting algorithm developed in an earlier paper. Computational experience of solving some fairly large problems by the proposed hybrid method is presented.

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